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LETTER TO THE EDITOR

The resistive susceptibility of random diode–insulator networks

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Abstract. We consider regular crystal lattices in which the bonds are either diodes (probability p) or insulators. The mean number of backbone bonds $L_{BB}(p)$ when averaged over lattice points is found to be related to the mean size of clusters $S(p)$ by $L_{BB}(p) = zpS(p)$ where z is the number of bonds directed away from any lattice site. Thus L_{BB} diverges at p_c with the mean size exponent γ. The resistive susceptibility $\chi_R(p)$ of Harris and Fisch is expanded graphically as a power series in p and it is found that term by term $\chi_R(p) = \Psi_R(p)[S(p)]^2$ where $\Psi_R(p)$ is obtained from $\chi_R(p)$ by ignoring contributions from nodal graphs. The above results are valid for bond and site dilution in any dimension. For bond dilution on the square lattice we have determined $\Psi_R(p)$ to p^{17} and Padé analysis of the resulting series for $\chi_R(p)$ shows that it diverges with exponent $\gamma_R = 3.654 \pm 0.017$. Using a scaling relation the exponent t for the conductivity of the infinite cluster is estimated to be $t = 0.75 \pm 0.02$.

The conductivity of random resistor networks near the percolation threshold has been actively studied since the early work of Last and Thouless (1971). They observed that the conductivity $\Sigma(p)$ as measured across the faces of a cubic sample vanishes continuously as the critical probability p_c is approached from above with critical exponent $t > 1$. The precise value of t is difficult to determine and estimates range over a wide band of values in both two and three dimensions (Abeles *et al* 1975, Derrida and Vannimenus 1982, Harris and Kirkpatrick 1977, Strayley 1977, Watson and Leath 1974). The above estimates were obtained using computer simulations and depend on statistical analysis of the conductivity of clusters which span the sample.

An indirect approach (Harris and Fisch 1977, Fisch and Harris 1978) to the problem is to obtain low density expansions in powers of p for the resistive 'susceptibility'. This is defined by

$$\chi_R(p) = \sum_{\boldsymbol{r}} R(\boldsymbol{r}, p) \tag{1}$$

where $R(\boldsymbol{r}, p)$ is the mean resistance between lattice sites in the same cluster which are separated by vector $\boldsymbol{r}$. The critical exponent γ_R defined by

$$\chi_R(p) \sim (p_c - p)^{-\gamma_R}, \qquad p \to p_c, \tag{2}$$

is then found by Padé approximant analysis and the exponent t is deduced from the scaling relation $t = \gamma_R - \gamma + (d-2)\nu$ (Harris and Fisch 1977). The coefficient of p^n may be found exactly by consideration of clusters with at most n bonds, whereas the

direct method for finding t entails the calculation of the resistance of large spanning clusters.

The random diode–insulator networks we shall consider here are regular lattices in which the nearest-neighbour bonds are either diodes with unit conductance (probability p) or insulators. The diodes are oriented so that the current flow in any one of them has a positive component in some 'preferred' direction. This model was first studied by Redner (1982) who discovered that in two dimensions the conductivity in the preferred direction vanishes at p_c (the critical probability for directed percolation) with a vertical tangent in contrast to the horizontal approach in the isotropic case. A Monte Carlo calculation with finite size scaling (Redner and Mueller 1982) gave $t = 0.60 \pm 0.10$. Arora *et al* (1983) used a similar technique but used a scaling function which involved the fraction $B(p)$ of 'backbone' bonds in the infinite cluster. They showed that

$$B(p) = p[P(p)]^2 \sim (p - p_c)^{2\beta} \tag{3}$$

where β is the exponent of the percolation probability $P(p)$. Using the value of β obtained by series methods (Blease 1977a) they obtained the estimate $t = 0.73 \pm 0.10$. Our result agrees well with the latter value.

In this letter we extend the low density expansion work of Fisch and Harris (1978) to the diode–insulator problem on the square lattice with random bond removal. The ordinary percolation exponents for this problem as obtained by series methods (Blease 1977a,b, De'Bell and Essam 1983a, b) are ten times more accurate than for isotropic percolation and a fairly definitive value of the exponent γ_R (and hence t) can therefore be expected. The reason for this improved accuracy is that directed percolation series may be obtained to much greater length because of the Markovian nature of the problem. We first present some theoretical considerations which show how the Markov property simplifies the calculation of $\chi_R(p)$ and leads to an exact relation between the mean number of backbone bonds $L_{BB}(p)$ (Coniglio 1982) and the mean cluster size $S(p)$. The latter relation is similar to that of equation (3) (Arora *et al* 1983). Our theoretical results apply to bond and site percolation on suitably directed regular lattices in any dimension.

We use the 'linked cluster' (Essam 1970) or 'cumulant' (Harris and Fisch 1977) method which expresses $R(\boldsymbol{r}, p)$ as a weighted sum over directed two-rooted graphs. These graphs are subgraphs of the lattice graph G_L whose vertices and edges are the sites and nearest-neighbour bonds of the lattice. Thus

$$R(\boldsymbol{r}, p) = \sum_{g \in \Gamma(\boldsymbol{r})} w(g) p^{e} \tag{4}$$

where $\Gamma(\boldsymbol{r})$ is the set of two-rooted subgraphs of the lattice graph G_L; the root points correspond to the sites $\mathbf{0}$ and $\boldsymbol{r}$. A factor p is included for each of the e random elements (bonds, sites or both; the site $\mathbf{0}$ will always be assumed present with probability one). The weight $w(g)$ is independent of the lattice graph G_L from which g was derived (Essam 1970) and may be obtained by calculating the function R for g itself. Thus for $g \in \Gamma(\boldsymbol{r})$

$$R(g, p) = \sum_{g' \subseteq g} w(g') p^{e'} \tag{5}$$

where

$$R(g, p) \equiv \langle \eta R \rangle = \sum_{g' \subseteq g} \eta(g') R(g') p^{e'} (1-p)^{e-e'} \tag{6}$$

and $\eta = 1$ or 0 depending on whether or not there is a directed path between the roots of g'; $R(g')$ is the resistance between the roots in the case that there is such a path (for the site problem only vertex subgraphs or section graphs must be included in the sum). The weight $w(g)$ may be either calculated from the coefficient of p^e in $R(g, p)$ or recursively by evaluating $R(g', 1) = R(g')$ for all the subgraphs of g. We use the latter method on the computer but the former is a useful theoretical tool. Thus it is clear that if g involves bonds which cannot carry current (dangling bonds) then $w(g) = 0$ and we henceforth restrict $\Gamma(\boldsymbol{r})$ to exclude such graphs. Now suppose that g is a series combination of g_1 and g_2 the junction of which is then a nodal point of g (these graphs are known as nodal). For such graphs $\langle \eta R \rangle = \langle \eta_1 R_1 \rangle \langle \eta_2 \rangle + \langle \eta_1 \rangle \langle \eta_2 R_2 \rangle$ and equating coefficients of p^e gives

$$w(g) = w(g_1)d(g_2) + d(g_1)w(g_2) \tag{7}$$

where the 'directed d-weight' $d(g)$ is the weight which arises on expanding the pair connectedness $C(g, p) = \langle \eta \rangle$. It is known (Arrowsmith and Essam 1977) that $d(g) = (-1)^c$ where c is the number of independent cycles in g. In common with d-weights for undirected percolation, which do not have unit modulus, the directed d-weights have the property $d(g) = d(g_1)d(g_2)$.

These properties of the weights will now be used to express $\chi_R(\boldsymbol{r}, p)$ in terms of non-nodal graphs only. We first consider the mean size expressed in terms of the pair connectedness (Essam 1972)

$$S(p) = \sum_{\boldsymbol{r}} C(\boldsymbol{r}, p) \tag{8}$$

where $C(\mathbf{0}, p) = 1$. Expanding in powers of p

$$S(p) = \sum_{\boldsymbol{r}} \sum_{g \in \Gamma(\boldsymbol{r})} d(g) p^e. \tag{9}$$

Equations (4)–(9) are valid for both resistor and diode problems. Now let $N(\boldsymbol{r})$ be the graphs in $\Gamma(\boldsymbol{r})$ which are non-nodal. Since any nodal member of $\Gamma(\boldsymbol{r})$ may be expressed as a series combination of a non-nodal graph $g' \in N(\boldsymbol{r}')$ followed by $g'' \in \Gamma(\boldsymbol{r} - \boldsymbol{r}')$ and all such combinations give rise to a distinct member of $\Gamma(\boldsymbol{r})$ (this is not true for undirected percolation) we have

$$\begin{aligned} S(p) &= 1 + \sum_{\boldsymbol{r} \neq \mathbf{0}} \sum_{\boldsymbol{r}' \neq \mathbf{0}} \sum_{g' \in N(\boldsymbol{r}')} \sum_{g'' \in \Gamma(\boldsymbol{r} - \boldsymbol{r}')} d(g') d(g'') p^{e'} p^{e''} \\ &= 1 + \sum_{\boldsymbol{r}' \neq \mathbf{0}} \sum_{g' \in N(\boldsymbol{r}')} d(g') p^{e'} \sum_{\boldsymbol{r}''} \sum_{g'' \in \Gamma(\boldsymbol{r}'')} d(g'') p^{e''} \\ &= 1 + S^N(p) S(p) \end{aligned} \tag{10}$$

where

$$S^N(p) = \sum_{\boldsymbol{r} \neq \mathbf{0}} \sum_{g \in N(\boldsymbol{r})} d(g) p^e \tag{11}$$

is the non-nodal part of $S(p)$. (Note that $\Gamma(\mathbf{0})$ contains only the graph which is a single vertex for which we must take $d = 1$ and $e = 0$.) A similar decomposition of (1) using (7) leads to

$$\chi_R(p) = \Psi_R(p) S(p) + S^N(p) \chi_R(p) \tag{12}$$

which combined with (10) gives

$$\chi_R(p) = \Psi_R(p) [S(p)]^2 \tag{13}$$

where $\Psi_R(p)$ is defined by (11) with d replaced by w. The average resistance $L_R(p)$ is defined by Coniglio (1982) as the ratio of χ_R to S and hence $L_R(p) = \Psi_R(p)S(p)$.

Coniglio defines several other effective lengths by replacing R in (6) in turn by μ, the length of the shortest path between the roots, τ, the mean number of steps in a SAW between the roots, and β, the number of backbone bonds. All these quantities satisfy the relation leading to (7) and hence give rise to an equation of type (13). Many other quantities may replace R, a further important one being the number λ of cutting bonds ('red' bonds of Pike and Stanley 1981).

In the case of hypercubic lattices μ and τ are both proportional to t, the component of $\boldsymbol{r}$ in the preferred direction, and hence L_μ and L_τ diverge with the index $\nu_\parallel$ of the parallel connectedness length. We now show that when $R = \beta$ the corresponding length $L_{BB}(p)$ diverges with the mean size exponent γ. Now

$$\beta(g') = \sum_i \nu_i(g') \tag{14}$$

where $\nu_i(g') = 1$ or 0 according to whether or not the ith bond of g is a backbone bond of g'. The average $\langle \eta \nu_i \rangle$ is therefore the probability that the ith bond is a backbone bond in a cluster connecting the roots. If g is non-nodal then for any bond i in g there must in general be a path in g between the roots which bypasses i. Since such a path is irrelevant in the calculation of $\langle \eta \nu_i \rangle$ this quantity and hence $\langle \eta \beta \rangle$ is a polynomial of degree $< e$ and hence $w_{BB}(g) = 0$. The only exception is the graph with a single edge for which $w_{BB}(g) = 1$, hence $\Psi_{BB}(p) = zp$, where z is the number of outward directed bonds from any vertex. This establishes that

$$L_{BB}(p) = zpS(p) \tag{15}$$

and hence that L_{BB} diverges with the mean size exponent γ.

Equation (13) has only been derived term by term in the p expansion. However, when R is replaced by β this expansion can be avoided. The idea is best illustrated by deriving a similar formula for the number of backbone sites. The probability that site $\boldsymbol{r}'$ belongs to the backbone of a finite cluster connecting $\boldsymbol{0}$ to $\boldsymbol{r}$ is $C(\boldsymbol{r}', p)C(\boldsymbol{r}-\boldsymbol{r}', p)$ where $C(\boldsymbol{r}, p)$ is the probability that there is a path from $\boldsymbol{0}$ to $\boldsymbol{r}$. The expected number of such sites is obtained by summing this probability over $\boldsymbol{r}'$ and a final sum over $\boldsymbol{r}$ gives the 'susceptibility' $\chi_{BS}(p)$; thus

$$\chi_{BS}(p) = \sum_{\boldsymbol{r}} \sum_{\boldsymbol{r}'} C(\boldsymbol{r}', p)C(\boldsymbol{r}-\boldsymbol{r}', p) = [S(p)]^2 \tag{16}$$

where we have used equation (8). A bond $(\boldsymbol{r}', \boldsymbol{r}'+\boldsymbol{e})$ belongs to the backbone of a cluster connecting $\boldsymbol{0}$ to $\boldsymbol{r}$ provided that there is a path from $\boldsymbol{0}$ to $\boldsymbol{r}'$ and a path from $\boldsymbol{r}'+\boldsymbol{e}$ to $\boldsymbol{r}$. Here $\boldsymbol{e}$ is a nearest-neighbour lattice vector in the direction of the lattice bond and if there are z of these, formula (16) may be extended to give

$$\chi_{BB}(p) = zp[S(p)]^2 \tag{17}$$

where the additional factor of p corresponds to the backbone bond itself. This result is equivalent to (15).

For the Bethe lattice R and β are equal since there are no parallel paths and hence $\chi_R = \chi_{BB}$. For this lattice $\boldsymbol{r}$ is not defined but the above arguments hold if $\boldsymbol{r}$ is replaced by the number of steps from the origin. We consider a rooted Bethe lattice with z bonds directed away from each site and only one bond directed into each site (except for the root). Let Q be the probability that a given branch from the origin is finite.

It has been shown (Essam 1972) that Q is the appropriate root of $pQ^z - Q + 1 - p = 0$. For $p < p_c$, $Q = 1$, but for $p > p_c$, $Q < 1$. We may use (16) and (17) even for $p > p_c$ with $S(p) = Q^z/(1 - zpQ^{z-1})$, the mean size of finite clusters, provided that a further factor Q^{-z} is included for χ_{BS} and a factor Q^{-1} for χ_{BB}. Thus χ_{BS}, χ_{BB} and χ_R all diverge with index $\gamma_{BB} = 2$ above and below p_c. Similar results are found for the undirected Bethe lattice and for this problem the probability that a bond belongs to the backbone of an infinite cluster has been given by Larson and Davis (1982).

We now turn to the calculation of γ_R for the square lattice bond problem. We have calculated the weights of all non-nodal subgraphs with ≤ 17 edges (there are 320 of these) and the resulting rational coefficients for $\Psi_R(p)$ are given in table 1. Using

Table 1. $\Psi_R(p) = 2p - 3p^4 - 9p^6 + \sum_{n=7}^{\infty} a_n p^n$.

n	a_n(Num)	a_n(Den)
7	49	5
8	−30	1
9	1180	23
10	−339 379	3220
11	964 792	4991
12	−264 012 168 01	708 222 90
13	729 493 492 967 797 79	996 541 334 766 90
14	−697 952 248 875 287 474 051 453	471 135 604 209 161 735 844
15	461 201 018 503 342 552 993 373 187 989 704 97	154 114 649 402 788 374 909 254 763 717 60
16	−458 266 765 391 778 701 509 349 790 480 482 765 624 364 421 751 77	774 970 106 110 236 567 383 758 269 404 478 202 342 414 9800
17	111 887 692 160 532 124 010 476 793 421 525 912 249 882 825 517 075 171 269 633 779 170 028 424 856 802 568 780 432 01	984 296 267 762 153 383 879 220 041 658 027 420 079 272 753 172 441 801 034 132 125 475 010 828 564 376 637 200

Table 2. $\chi_R(p) = \sum_{n=1}^{\infty} b_n p^n$.

n	b_n
1	0.200 000 000 000 000 000 000 000 00D+01
2	0.800 000 000 000 000 000 000 000 00D+01
3	0.240 000 000 000 000 000 000 000 00D+02
4	0.610 000 000 000 000 000 000 000 00D+02
5	0.144 000 000 000 000 000 000 000 00D+03
6	0.315 000 000 000 000 000 000 000 00D+03
7	0.669 800 000 000 000 000 000 000 00D+03
8	0.135 520 000 000 000 000 000 000 00D+04
9	0.271 090 434 782 608 695 652 173 91D+04
10	0.522 742 018 633 540 372 670 807 45D+04
11	0.100 397 697 054 698 457 223 001 40D+05
12	0.187 224 168 970 390 536 651 667 15D+05
13	0.349 664 053 941 286 075 394 551 40D+05
14	0.635 311 056 113 312 955 180 067 16D+05
15	0.116 318 108 711 560 621 780 167 41D+06
16	0.207 078 081 209 702 271 299 567 18D+06
17	0.373 220 805 502 454 694 856 887 21D+06

the known series for $S(p)$ (Blease 1977b) we have obtained $\chi_R(p)$ and the coefficients, rounded to 26 digits, are given in table 2. To this order in p no negative flow was encountered in the calculation of R. The usual Padé analysis gives rise to the pole–residue plot shown in figure 1. From this we conclude that

$$\gamma_R = 3.654 + 70\Delta p_c \pm 0.003 \tag{18}$$

where the central value corresponds to taking $p_c = 0.6446$ (Blease 1977b, De'Bell and Essam 1983a); the uncertainty in this value is 0.0002.

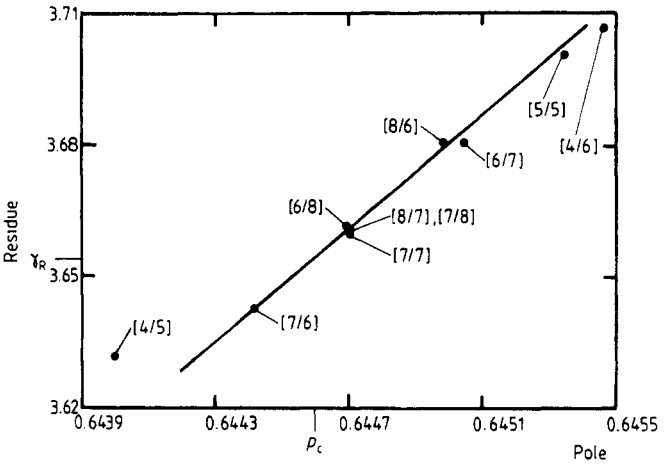

Figure 1. Pole–residue plot for Padé approximants to $d \log \chi_R(p)$.

Redner (1982) has extended the scaling formula (3) to the diode problem and finds

$$t = \zeta_R + (d-1)\nu_\perp - \nu_\parallel. \tag{19}$$

With $\zeta_R = \gamma_R - \gamma$ and using the results of De'Bell and Essam (1983a) for γ, $\nu_\parallel$ and $\nu_\perp$ we obtain

$$t = 0.750 - 30\Delta p_c \pm 0.012. \tag{20}$$

Taking into account the uncertainty in p_c yields the results quoted in the abstract.

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